

# ON THE HARDY–LITTLEWOOD MAJORANT PROBLEM FOR ARITHMETIC SETS

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ABSTRACT. The aim of this paper is to exhibit a wide class of sparse deterministic sets,  $\mathbf{B} \subseteq \mathbb{N}$ , so that

$$\limsup_{N \rightarrow \infty} N^{-1} |\mathbf{B} \cap [1, N]| = 0,$$

for which the Hardy–Littlewood majorant property holds:

$$\sup_{|a_n| \leq 1} \left\| \sum_{n \in \mathbf{B} \cap [1, N]} a_n e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)} \leq \mathbf{C}_p \left\| \sum_{n \in \mathbf{B} \cap [1, N]} e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)},$$

where  $p \geq p_{\mathbf{B}}$  is sufficiently large, the implicit constant  $\mathbf{C}_p$  is independent of  $N$ , and the supremum is taken over all complex sequences  $(a_n : n \in \mathbb{N})$  such that  $|a_n| \leq 1$ .

## 1. INTRODUCTION

In 1937, Hardy and Littlewood [7] conjectured that for each  $p \geq 2$  there is a constant  $\mathbf{C}_p > 0$  such that for every finite set  $A \subset \mathbb{N}$  and every sequence  $(a_n : n \in A)$  of complex numbers satisfying  $\sup_{n \in A} |a_n| \leq 1$  we have

$$(1) \quad \left\| \sum_{n \in A} a_n e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)} \leq \mathbf{C}_p \left\| \sum_{n \in A} e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)}.$$

This conjecture, known as the *Hardy–Littlewood majorant problem*, was suggested by a simple observation, based on Parseval’s identity, which implies that  $\mathbf{C}_p = 1$  for every even integer  $p \geq 2$ . It was also noticed by Hardy and Littlewood that  $\mathbf{C}_3 > 1$ . In 1962, Boas [2] showed that  $\mathbf{C}_p > 1$  for any  $p \notin \{2k : k \in \mathbb{N}\}$ . Finally, in early seventies Bachelis [1] disproved the Hardy–Littlewood conjecture showing unboundedness of  $\mathbf{C}_p$  for every  $p \notin \{2k : k \in \mathbb{N}\}$  as  $|A| \rightarrow \infty$ .

Although inequality (1) fails to hold in general, recently some attention has been paid to quantify this failure. To do so, for  $N \in \mathbb{N}$  we consider

$$\mathbf{C}_p(N) = \sup_{A \subseteq \{1, \dots, N\}} \mathbf{C}_p(A, N)$$

where for  $A \subseteq \{1, \dots, N\}$  we have set

$$\mathbf{C}_p(A, N) = \sup_{|a_n| \leq 1} \left\| \sum_{n \in A} a_n e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)} \cdot \left\| \sum_{n \in A} e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)}^{-1}.$$

It was proven in [10] that for every  $p \in (2, 4)$  there is a constant  $C > 0$  such that

$$\log \mathbf{C}_p(N) \geq C \frac{\log N}{\log \log N}.$$

Consequently, the Hardy–Littlewood majorant problem was reformulated to a slightly weaker statement. Namely, it was conjectured that for every  $p \geq 2$  and  $\varepsilon > 0$  there is a constant  $C_{p, \varepsilon} > 0$  such that for every  $N \in \mathbb{N}$

$$(2) \quad \mathbf{C}_p(N) \leq C_{p, \varepsilon} N^\varepsilon.$$

It is worth mentioning that (2) implies the restriction conjecture for the Fourier transform on  $\mathbb{R}^d$ , i.e. that for every  $p > 2d/(d-1)$  there exists a constant  $C_{p,d} > 0$  such that

$$(3) \quad \|\widehat{f d\sigma}\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \|f\|_{L^\infty(\mathbb{S}^{d-1}, d\sigma)}$$

where  $\sigma$  is the spherical measure on the unit sphere  $\mathbb{S}^{d-1}$  in  $\mathbb{R}^d$ . In [10] it was stated that for suitable sets  $A$  the inequality (1) may be treated as a restatement of (3). However, Mockenhaupt and Schlag [11] disproved (2) by showing that for all  $p > 2$  which is not an even integer, there are constants  $\eta > 0$  and  $C > 0$  such that  $\mathbf{C}_p(N) \geq CN^\eta$ . For  $p = 3$  the same result was obtained by Green and Ruzsa [5].

In view of the restriction conjecture one may ask whether there are sets  $A \subseteq \{1, \dots, N\}$  such that for every  $p \geq 2$  and  $\varepsilon > 0$  there exists a constant  $C_{p,\varepsilon} > 0$  for which we have

$$(4) \quad \mathbf{C}_p(A, N) \leq C_{p,\varepsilon} N^\varepsilon.$$

The question above has been extensively studied by Mockenhaupt and Schlag in [11] where the authors proved that for every  $\varrho \in (0, 1)$  and  $p \geq 2$  there are random sets  $A \subseteq \{1, \dots, N\}$  with cardinality  $N^\varrho$  satisfying (4) with a large probability.

The Hardy–Littlewood majorant property plays an important role in combinatorial problems. In [4] Green used a variant of the inequality (1) for the set of prime numbers  $\mathbb{P}$  to deduce that every subset of  $\mathbb{P}$  with non-vanishing relative upper-density contains at least one arithmetic progression of length three. Specifically, Green proved that for every  $p \geq 2$  there is a constant  $C_p > 0$  such that for all  $N \in \mathbb{N}$

$$\mathbf{C}_p(\mathbb{P}_N, N) \leq C_p$$

where  $\mathbb{P}_N = \mathbb{P} \cap [1, N]$ , the set of primes less than or equal to  $N$ . Generally speaking, in problems of this kind it is critical to know whether the majorant property (1) holds for some  $p \in (2, 3)$  with the uniform constant  $\mathbf{C}_p$ , independent of the cardinality of the set  $A$  (see [6, 12]).

The present article is devoted to study a wide class of deterministic infinite sets  $A \subseteq \mathbb{N}$  with vanishing Banach density, i.e.

$$\limsup_{N \rightarrow \infty} \frac{|A \cap [1, N]|}{N} = 0,$$

and obeying the Hardy–Littlewood majorant property. In particular, we will be concerned with the sets

$$(5) \quad \mathbf{A} = \{ \lfloor h(n) \rfloor : n \in \mathbb{N} \}$$

where  $h$  is a regularly varying function of the form  $h(x) = x\ell(x)$ , for a suitably chosen slowly varying function  $\ell$ , e.g.

$$\ell(x) = (\log x)^B, \quad \text{or} \quad \ell(x) = \exp(B(\log x)^C), \quad \text{or} \quad \ell(x) = l_m(x),$$

where  $B > 0$ ,  $C \in (0, 1)$ ,  $l_1(x) = \log x$  and  $l_{m+1}(x) = \log(l_m(x))$ , for  $m \in \mathbb{N}$ . We show that for every  $p \geq 2$  there exists a constant  $C_p > 0$  such that for every  $N \in \mathbb{N}$  we have

$$\mathbf{C}_p(\mathbf{A}_N, N) \leq C_p$$

where  $\mathbf{A}_N = \mathbf{A} \cap [1, N]$ . We also consider the sets (5) with

$$h(x) = x^c \ell(x)$$

for some  $c > 1$  sufficiently close to 1. In this case we show that it is possible to find  $p_c > 2$  such that for every  $p > p_c$  there exists a constant  $C_{c,p} > 0$  such that for every  $N \in \mathbb{N}$

$$\mathbf{C}_p(\mathbf{A}_N, N) \leq C_{c,p}.$$

Moreover,  $\lim_{c \rightarrow 1} p_c = 2$ .

**1.1. Statement of the results.** Before we precisely formulate the main results we need to introduce some definitions.

**Definition 1.1.** Let  $\mathcal{L}$  be a family of slowly varying functions  $\ell : [x_0, \infty) \rightarrow (0, \infty)$  such that

$$\ell(x) = \exp \left( \int_{x_0}^x \frac{\vartheta(t)}{t} dt \right)$$

where  $\vartheta \in \mathcal{C}^2([x_0, \infty))$  is a real function satisfying

$$\lim_{x \rightarrow \infty} \vartheta(x) = 0, \quad \lim_{x \rightarrow \infty} x\vartheta'(x) = 0, \quad \lim_{x \rightarrow \infty} x^2\vartheta''(x) = 0.$$

We also distinguish a subfamily  $\mathcal{L}_0$  of  $\mathcal{L}$ .

**Definition 1.2.** Let  $\mathcal{L}_0$  be a family of slowly varying functions  $\ell : [x_0, \infty) \rightarrow (0, \infty)$  such that

$$\ell(x) = \exp \left( \int_{x_0}^x \frac{\vartheta(t)}{t} dt \right)$$

where  $\vartheta \in \mathcal{C}^2([x_0, \infty))$  is positive decreasing real function satisfying

$$\lim_{x \rightarrow \infty} \vartheta(x) = 0, \quad \lim_{x \rightarrow \infty} \frac{x\vartheta'(x)}{\vartheta(x)} = 0, \quad \lim_{x \rightarrow \infty} \frac{x^2\vartheta''(x)}{\vartheta(x)} = 0,$$

and for every  $\varepsilon > 0$  there is a constant  $C_\varepsilon > 0$  such that  $1 \leq C_\varepsilon \vartheta(x)x^\varepsilon$  and  $\lim_{x \rightarrow \infty} \ell(x) = \infty$ .

Finally, we define the subfamily  $\mathcal{R}_c$  of regularly varying functions.

**Definition 1.3.** For every  $c \in (0, 2) \setminus \{1\}$  let  $\mathcal{R}_c$  be a family of increasing, convex, regularly-varying functions  $h : [x_0, \infty) \rightarrow [1, \infty)$  of the form

$$h(x) = x^c L(x)$$

where  $L \in \mathcal{L}$ . If  $c = 1$  we impose that  $L \in \mathcal{L}_0$ .

We fix two functions  $h_1 \in \mathcal{R}_{c_1}$  and  $h_2 \in \mathcal{R}_{c_2}$  for  $c_1 \in [1, 2)$  and  $c_2 \in [1, 6/5)$ . Let  $\varphi_1$  and  $\varphi_2$  be the inverse of  $h_1$  and  $h_2$ , respectively. We consider a function  $\psi : [x_0, \infty) \rightarrow (0, \infty)$  such that for all  $x \geq x_0$ ,  $\psi(x) \leq 1/2$  and

$$(6) \quad \lim_{x \rightarrow +\infty} \frac{\psi(x)}{\varphi_2'(x)} = 1, \quad \lim_{x \rightarrow +\infty} \frac{\psi'(x)}{\varphi_2''(x)} = 1, \quad \lim_{x \rightarrow +\infty} \frac{\psi''(x)}{\varphi_2'''(x)} = 1.$$

Finally, we define two sets

$$\mathbf{B}_+ = \{n \in \mathbb{N} : \{\varphi_1(n)\} < \psi(n)\}, \quad \mathbf{B}_- = \{n \in \mathbb{N} : \{-\varphi_1(n)\} < \psi(n)\}.$$

Let us observe that if  $h_1 = h_2 = h$  is the inverse function  $\varphi$  and  $\psi(x) = \varphi(x+1) - \varphi(x)$  then  $\mathbf{B}_- = \mathbf{A}$ . Indeed, we have the following chain of equivalences

$$\begin{aligned} m \in \mathbf{A} &\iff m = \lfloor h(n) \rfloor \text{ for some } n \in \mathbb{N} \\ &\iff h(n) - 1 < m \leq h(n) < m + 1 \\ &\iff \varphi(m) \leq n < \varphi(m+1), \text{ since } \varphi \text{ is well-defined and monotonically increasing} \\ &\iff 0 \leq n - \varphi(m) < \varphi(m+1) - \varphi(m) = \psi(m) < 1/2 \\ &\iff 0 \leq \{-\varphi(m)\} < \psi(m) \\ &\iff m \in \mathbf{B}_-. \end{aligned}$$

The main result of this paper is the following theorem.

**Theorem 1.** *Assume that  $c_1 \in [1, 2)$  and  $c_2 = 1$ . Then for every  $p \geq 2$  there exists a constant  $\mathbf{C}_p > 0$  such that for every  $N \in \mathbb{N}$  and any sequence of complex numbers  $(a_n : n \in \mathbb{N})$  satisfying  $\sup_{n \in \mathbb{N}} |a_n| \leq 1$  we have*

$$(7) \quad \left\| \sum_{n \in \mathbf{B}_N^\pm} a_n e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)} \leq \mathbf{C}_p \left\| \sum_{n \in \mathbf{B}_N^\pm} e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)}$$

where  $\mathbf{B}_N^\pm = \mathbf{B}_\pm \cap [1, N]$ .

We observe that by the Hausdorff–Young inequality for every  $p \geq 2$  we obtain

$$\left\| \sum_{n \in \mathbf{B}_N^\pm} a_n e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)} \leq |\mathbf{B}_N^\pm|^{1/p'}.$$

Moreover, by integrating over frequencies  $|\xi| \leq 1/(100N)$ , we have the following lower bound

$$\left\| \sum_{n \in \mathbf{B}_N^\pm} e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)} \gtrsim |\mathbf{B}_N^\pm| N^{-1/p}.$$

These inequalities combined together yield

$$(8) \quad \left\| \sum_{n \in \mathbf{B}_N^\pm} a_n e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)} \lesssim |\mathbf{B}_N^\pm|^{1/p} N^{-1/p} \left\| \sum_{n \in \mathbf{B}_N^\pm} e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)}.$$

By Proposition 2.1 for  $c_2 = 1$ , we have  $|\mathbf{B}_N^\pm| \sim \varphi_2(N)$  where  $\varphi_2(N) = NL_{\varphi_2}(N)$  for some slowly varying function  $L_{\varphi_2} \in \mathcal{L}_0$ . Therefore, applying inequality (8), we obtain

$$\mathbf{C}_p(\mathbf{B}_N^\pm, N) \lesssim L_{\varphi_2}(N)^{1/p} \lesssim N^\varepsilon$$

for any  $\varepsilon > 0$ . Hence, the main difficulty in proving Theorem 1 is to show that the constant in (7) is independent of  $N$ .

Next, we would like to relax the hypothesis in Theorem 1 to allow any  $c_2 \in [1, 6/5)$ . It is possible at the expense of a slightly worse range of  $p$ . Let us introduce

$$p(c_1, c_2) = \frac{2/c_1 - 6/c_2 + 6}{1/c_1 + 3/c_2 - 3} = 2 + \frac{12 - 12/c_2}{1/c_1 + 3/c_2 - 3}.$$

We observe that if  $c_1 \in [1, 2)$  and  $c_2 \in [1, 6/5)$  then  $1 < \frac{1}{3c_1} + \frac{1}{c_2}$ , thus

$$\frac{12 - 12/c_2}{1/c_1 + 3/c_2 - 3} \geq 0.$$

Also notice that

$$\lim_{c_2 \rightarrow 1} p(c_1, c_2) = 2.$$

The extended version of Theorem 1 has the following form.

**Theorem 2.** *Assume that  $c_1 \in [1, 2)$  and  $c_2 \in [1, 6/5)$ . Then for every  $p \geq p(c_1, c_2)$  there exists a constant  $\mathbf{C}_p > 0$  such that for every  $N \in \mathbb{N}$  and any sequence of complex numbers  $(a_n : n \in \mathbb{N})$  satisfying  $\sup_{n \in \mathbb{N}} |a_n| \leq 1$  we have*

$$\left\| \sum_{n \in \mathbf{B}_N^\pm} a_n e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)} \leq \mathbf{C}_p \left\| \sum_{n \in \mathbf{B}_N^\pm} e^{2\pi i n \xi} \right\|_{L^p(\mathbb{T}, d\xi)}$$

where  $\mathbf{B}_N^\pm = \mathbf{B}_\pm \cap [1, N]$ .

We were inspired to study Hardy–Littlewood majorant property by the paper of Mockenhaupt and Schlag [11] where the authors considered sparse random subsets of the integers. The desire to better understand structure of deterministic sets which satisfy (1) was our principal motivation.

Before turning to the arguments, let us begin with some preliminary remarks. The heart of the matter lies in proving our Proposition 3.1, which can be thought of as a restriction estimate for our sets  $\mathbf{B}_N^\pm$ . We accomplish this using a Tomas–Stein  $TT^*$  argument, which forces us to estimate certain exponential sums, see Section 3 below. These estimates are quite delicate, and lead to the technical restriction on the range of  $L^p$  spaces which we are able to handle; in particular, we do not yet know how to extend Theorem 2 to the full regime  $2 < p < p(c_1, c_2)$ . Finally, it is worth calling attention to the explicit construction of the sets  $\mathbf{B}_N^\pm$  for which the full strength of the Hardy–Littlewood property holds. To the best of the authors knowledge it is the first treatment where such a wide family of subsets of the integers satisfies property (1).

## 2. SOME PROPERTIES OF THE SETS $\mathbf{B}_\pm$

As it has been observed, when  $c_1 \in [1, 2)$  and  $c_2 \in [1, 6/5)$  we have  $1 < \frac{1}{3c_1} + \frac{1}{c_2}$ , or equivalently

$$3(1 - \gamma_2) + (1 - \gamma_1) < 1,$$

where  $\gamma_1 = 1/c_1$  and  $\gamma_2 = 1/c_2$ . Under this assumption, we prove the asymptotic formula for the cardinality of sets  $\mathbf{B}_\pm$ .

**Proposition 2.1.** *For every  $\epsilon > 0$*

$$(9) \quad |\mathbf{B}_N^\pm| = \varphi_2(N)(1 + \mathcal{O}(N^{-\epsilon})).$$

From now on we only work with the sets  $\mathbf{B}_+$  because all the results remain valid for  $\mathbf{B}_-$  with similar proofs. To simplify the notation we write

$$\mathbf{B} = \mathbf{B}_+ = \{n \in \mathbb{N} : \{\varphi_1(n)\} < \psi(n)\}.$$

We need the following working characterizations of the sets  $\mathbf{B}$ .

**Lemma 2.2.**  *$n \in \mathbf{B}$  if and only if  $\lfloor \varphi_1(n) \rfloor - \lfloor \varphi_1(n) - \psi(n) \rfloor = 1$ .*

*Proof.* We begin with the forward implication; it suffices to show that if  $n \in \mathbf{B}$ , the integer

$$\lfloor \varphi_1(n) \rfloor - \lfloor \varphi_1(n) - \psi(n) \rfloor$$

belongs to  $(0, 3/2)$ . By definition, if  $n \in \mathbf{B}$  then  $0 \leq \varphi_1(n) - \lfloor \varphi_1(n) \rfloor < \psi(n)$ , thus

$$-\varphi_1(n) \leq -\lfloor \varphi_1(n) \rfloor < \psi(n) - \varphi_1(n)$$

if and only if

$$\varphi_1(n) \geq \lfloor \varphi_1(n) \rfloor > \varphi_1(n) - \psi(n),$$

from where it follows that

$$\lfloor \varphi_1(n) \rfloor - \lfloor \varphi_1(n) - \psi(n) \rfloor > \{\varphi_1(n) - \psi(n)\} \geq 0.$$

In view of  $\lfloor \varphi_1(n) - \psi(n) \rfloor \geq \varphi_1(n) - \psi(n) - 1$ , we obtain

$$\begin{aligned} \lfloor \varphi_1(n) \rfloor - \lfloor \varphi_1(n) - \psi(n) \rfloor &\leq \lfloor \varphi_1(n) \rfloor - \varphi_1(n) + \psi(n) + 1 \\ &\leq \psi(n) + 1 < 3/2. \end{aligned}$$

We now turn to the reverse implication; if  $\lfloor \varphi_1(n) \rfloor = 1 + \lfloor \varphi_1(n) - \psi(n) \rfloor$ , we have

$$\begin{aligned} 0 \leq \varphi_1(n) - \lfloor \varphi_1(n) \rfloor &= \varphi_1(n) - 1 - \lfloor \varphi_1(n) - \psi(n) \rfloor \\ &< \varphi_1(n) - 1 + 1 + \psi(n) - \varphi_1(n) = \psi(n). \end{aligned}$$

Consequently, we get  $\{\varphi_1(n)\} < \psi(n)$ , as desired.  $\square$

Our next task is to show that for every  $\delta \geq 0$  satisfying  $3(1 - \gamma_2) + (1 - \gamma_1) + 6\delta < 1$  there is  $\delta' > 0$  such that

$$(10) \quad \sum_{n \in \mathbf{B}_N} e^{2\pi i \xi n} = \sum_{n=1}^N \psi(n) e^{2\pi i \xi n} + \mathcal{O}(\varphi_2(N) N^{-\delta-\delta'})$$

where the implied constant is independent of  $\xi$  and  $N$ . Let us observe that the asymptotic formula (9) follows from (10) by taking  $\xi = 0$ . Indeed, we have

$$|\mathbf{B}_N| = \sum_{n \in \mathbf{B}_N} 1 = \sum_{n=1}^N \psi(n) + \mathcal{O}(\varphi_2(N) N^{-\varepsilon})$$

and summation by parts yields

$$(11) \quad \frac{1}{\varphi_2(N)} \sum_{n=1}^N \psi(n) = \frac{N\psi(N)}{\varphi_2(N)} - \frac{1}{\varphi_2(N)} \int_1^N x\psi'(x) dx = \frac{1}{\varphi_2(N)} \int_1^N \psi(x) dx = 1 + o(1).$$

Although, for the proof of (9) we only needed (10) with  $\xi = 0$ , the more general version will be used in our future works.

For the proof of (10), let us introduce the “sawtooth” function  $\Phi(x) = \{x\} - 1/2$ . Notice that

$$[\varphi_1(n)] - [\varphi_1(n) - \psi(n)] = \psi(n) + \Phi(\varphi_1(n) - \psi(n)) - \Phi(\varphi_1(n)).$$

With this in mind, we may write

$$(12) \quad \sum_{n \in \mathbf{B}_N} e^{2\pi i \xi n} = \sum_{n=1}^N \psi(n) e^{2\pi i \xi n} + \sum_{n=1}^N (\Phi(\varphi_1(n) - \psi(n)) - \Phi(\varphi_1(n))) e^{2\pi i \xi n}.$$

The second sum we absorb into an error term of the order  $\mathcal{O}(\varphi_2(N) N^{-\varepsilon})$ . To do so, see [8], we expand  $\Phi$  into its Fourier series, i.e.

$$\Phi(x) = \sum_{0 < |m| \leq M} \frac{1}{2\pi i m} e^{-2\pi i m x} + \mathcal{O}\left(\min\left\{1, \frac{1}{M\|x\|}\right\}\right),$$

for some  $M > 0$  where  $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$  is the distance of  $x \in \mathbb{R}$  to the nearest integer. Next, we expand

$$(13) \quad \min\left\{1, \frac{1}{M\|x\|}\right\} = \sum_{m \in \mathbb{Z}} b_m e^{2\pi i m x}$$

where

$$(14) \quad |b_m| \lesssim \min\left\{\frac{\log M}{M}, \frac{1}{|m|}, \frac{M}{|m|^2}\right\}.$$

We split the second sum in (12) into three parts,

$$\begin{aligned} I_1 &= \sum_{0 < |m| \leq M} \frac{1}{2\pi i m} \sum_{n=1}^N e^{2\pi i (n\xi - m\varphi_1(n))} (e^{2\pi i m \psi(n)} - 1), \\ I_2 &= \mathcal{O}\left(\sum_{n=1}^N \min\left\{1, \frac{1}{M\|\varphi_1(n) - \psi(n)\|}\right\}\right), \\ I_3 &= \mathcal{O}\left(\sum_{n=1}^N \min\left\{1, \frac{1}{M\|\varphi_1(n)\|}\right\}\right). \end{aligned}$$

Now, our aim is to show that each part  $I_1, I_2$  and  $I_3$  is  $\mathcal{O}(\varphi_2(N)N^{-\varepsilon})$ . In the proof we use the estimates for the following trigonometric sums: for  $m \in \mathbb{Z} \setminus \{0\}$ ,  $l \in \{0, 1\}$  and  $X \leq X' \leq 2X$  we consider

$$\sum_{X \leq n \leq X' \leq 2X} e^{2\pi i(\xi n + m(\varphi_1(n) - l\psi(n)))}$$

By [9, Lemma 2.14], if  $c_1 = 1$  then there is a positive decreasing real function  $\sigma_1$  satisfying  $\sigma_1(2x) \simeq \sigma_1(x)$  and  $\sigma_1(x) \gtrsim x^{-\varepsilon}$  for any  $\varepsilon > 0$ , such that

$$(15) \quad \varphi_1''(x) \simeq \frac{\varphi_1(x)\sigma_1(x)}{x^2}.$$

We set  $\sigma_1 \equiv 1$  whenever  $c_1 > 1$ . Similarly, by [9, Lemma 2.14] for  $\varphi_2'''$  we obtain

$$(16) \quad \varphi_2'''(x) \simeq \frac{\varphi_2(x)}{x^3}.$$

Therefore, by (6) we may write

$$|\psi''(x)| \simeq |\varphi_2'''(x)| \simeq \frac{\varphi_2(x)}{x^3}.$$

Since  $1/c_2 \leq 1 < 1 + 1/c_1$ , we get

$$\frac{\varphi_2(x)}{x\sigma_1(x)\varphi_1(x)} = o(1),$$

thus

$$|\psi''(x)| = o\left(\frac{\sigma_1(x)\varphi_1(x)}{x^2}\right).$$

Let  $F(x) = \xi x + m(\varphi_1(x) - l\psi(x))$ . By (15) and (16), for any  $X \leq x \leq X' \leq 2X$  we may write

$$|F''(x)| = |m| \cdot |\varphi_1''(x) - l\psi''(x)| \simeq \frac{|m|\sigma_1(X)\varphi_1(X)}{X^2}.$$

Therefore, the Van der Corput lemma (see [3, Theorem 2.2]) yields

$$(17) \quad \left| \sum_{X < n \leq 2X} e^{2\pi i(\xi n + m(\varphi_1(n) - l\psi(n)))} \right| \lesssim X \left( \frac{m\sigma_1(X)\varphi_1(X)}{X^2} \right)^{1/2} + \left( \frac{X^2}{m\sigma_1(X)\varphi_1(X)} \right)^{1/2} \\ \lesssim m^{1/2} X (\sigma_1(X)\varphi_1(X))^{-1/2}.$$

Finally, we get

$$\left| \sum_{j=1}^N e^{2\pi i(\xi n + m(\varphi_1(n) - l\psi(n)))} \right| \leq \sum_{j=0}^{\lceil \log N \rceil} \left| \sum_{\substack{2^j < n \leq 2^{j+1} \\ n \leq N}} e^{2\pi i(\xi n + m(\varphi_1(n) - l\psi(n)))} \right| \\ \lesssim m^{1/2} N (\log N) (\sigma_1(N)\varphi_1(N))^{-1/2}$$

since the function  $x \mapsto x(\sigma_1(x)\varphi_1(x))^{-1/2}$  is increasing. In particular, we have proven the following lemma.

**Lemma 2.3.** *There is a positive decreasing real function  $\sigma_1$  satisfying  $\sigma_1(2x) \simeq \sigma_1(x)$  and  $\sigma_1(x) \gtrsim x^{-\varepsilon}$ , for any  $\varepsilon > 0$ , such that for every  $m \in \mathbb{Z} \setminus \{0\}$ ,  $l \in \{0, 1\}$ , and  $N \geq 1$  we have*

$$\left| \sum_{1 \leq n \leq N} e^{2\pi i(\xi n + m(\varphi_1(n) - l\psi(n)))} \right| \lesssim |m|^{1/2} N (\log N) (\sigma_1(N)\varphi_1(N))^{-1/2}.$$

If  $c_1 > 1$  then  $\sigma_1 \equiv 1$ . The implied constant is independent of  $m$ ,  $N$  and  $\xi$ .

Next, we return to bounding  $I_1, I_2$  and  $I_3$ .

**2.1. The estimate for  $I_1$ .** Let

$$S(x) = \sum_{x \leq n \leq x' < 2x} e^{2\pi i(n\xi - m\varphi_1(n))}$$

and  $\phi_m(x) = e^{2\pi i m\psi(x)} - 1$ . We observe that

$$|\phi_m(x)| \lesssim mx^{-1}\varphi_2(x)$$

and

$$|\phi'_m(x)| \lesssim mx^{-2}\varphi_2(x).$$

Applying to the inner sum in  $I_1$  summation by parts together with (17) we obtain

$$\begin{aligned} & \left| \sum_{n=1}^N e^{2\pi i(n\xi - m\varphi_1(n))} \phi_m(n) \right| \\ & \leq (\log N) \sup_{X \in [1, N]} \left( |S(2X)| \cdot |\phi_m(2X)| + |S(X)| \cdot |\phi_m(X)| + \int_X^{2X} |S(x)| \cdot |\phi'_m(x)| dx \right) \\ & \lesssim (\log N) \sup_{X \in [1, N]} m^{3/2} \varphi_2(X) (\sigma_1(X) \varphi_1(X))^{-1/2} \\ & \leq m^{3/2} \varphi_2(N) (\log N) (\sigma_1(N) \varphi_1(N))^{-1/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} |I_1| & \lesssim \sum_{m=1}^M m^{1/2} (\log N) \varphi_2(N) (\sigma_1(N) \varphi_1(N))^{-1/2} \\ & \lesssim M^{3/2} (\log N) \varphi_2(N) (\sigma_1(N) \varphi_1(N))^{-1/2}. \end{aligned}$$

**2.2. The estimates for  $I_2$  and  $I_3$ .** We only treat  $I_2$  because  $I_3$  can be handled by a similar reasoning. By (13), (14) and Lemma 2.3 we have

$$\begin{aligned} & \sum_{n=1}^N \min \left\{ 1, \frac{1}{M \|\varphi_1(n) - \psi(n)\|} \right\} \leq \sum_{m \in \mathbb{Z}} |b_m| \left| \sum_{n=1}^N e^{2\pi i m(\varphi_1(n) - \psi(n))} \right| \\ & \lesssim \frac{N(\log M)}{M} + \left( \sum_{0 < |m| \leq M} \frac{\log M}{M} + \sum_{|m| > M} \frac{M}{|m|^2} \right) |m|^{1/2} (\log N) \varphi_2(N) (\sigma_1(N) \varphi_1(N))^{-1/2} \\ & \lesssim \frac{N(\log M)}{M} + M^{1/2} (\log M) (\log N) \varphi_2(N) (\sigma_1(N) \varphi_1(N))^{-1/2}. \end{aligned}$$

**2.3. Concluding remarks.** Based on Subsections 2.1 and 2.2 we get

$$|I_1| + |I_2| + |I_3| \lesssim \frac{N \log M}{M} + M^{3/2} (\log M) (\log N) \varphi_2(N) (\sigma_1(N) \varphi_1(N))^{-1/2}.$$

Therefore, by taking  $M = N^{1+\delta} (\log N) \varphi_2(N)^{-1}$ , we conclude

$$\begin{aligned} |I_1| + |I_2| + |I_3| & \lesssim \varphi_2(N) N^{-\delta} \left( 1 + N^{3/2+5\delta/2} (\log N) \sigma_1(N)^{-1/2} \varphi_1(N)^{-1/2} \varphi_2(N)^{-3/2} \right) \\ & \lesssim \varphi_2(N) N^{-\delta} \left( 1 + N^{3/2+6\delta/2-\gamma_1/2-3\gamma_2/2} \right) \end{aligned}$$

which is bounded by a constant multiple of  $\varphi_2(N) N^{-\delta}$  since  $3(1 - \gamma_2) + (1 - \gamma_1) + 6\delta < 1$ .



## 3. PROOF OF THEOREM 1

Let

$$\mathcal{F}(f)(\xi) = \sum_{n \in \mathbb{Z}} f(n) e^{2\pi i \xi n}$$

denote the Fourier transform on  $\mathbb{Z}$ , and

$$\hat{f}(n) = \int_{\mathbb{T}} f(\xi) e^{-2\pi i \xi n} d\xi$$

denote the Fourier transform on  $\mathbb{T}$ . For any measure space  $X$ , let  $\mathcal{C}(X)$  be the space of all continuous functions on  $X$ . For  $N \in \mathbb{N}$  we introduce on  $\mathbb{Z}$  a measure  $\mu_N$  defined

$$\mu_N(x) = N^{-1} \sum_{n \in \mathbf{B}_N} \psi(n)^{-1} \delta_n(x).$$

Let  $T_N : \mathcal{C}(\mathbf{B}_N) \rightarrow \mathcal{C}(\mathbb{T})$  be the linear operator given by

$$T_N(f) = \mathcal{F}(f\mu_N).$$

We are going to prove the following proposition.

**Proposition 3.1.** *For each*

$$(18) \quad p \geq 2 + \frac{12 - 12/c_2}{1/c_1 + 3/c_2 - 3}$$

*there is a constant  $C_p > 0$  such that for all  $N \in \mathbb{N}$  and  $f \in L^2(\mathbf{B}_N, \mu_N)$*

$$\|T_N f\|_{L^p(\mathbb{T})} \leq C_p N^{-1/p} \|f\|_{L^2(\mathbf{B}_N, \mu_N)}.$$

Before embarking on the proof we show the following.

**Lemma 3.2.** *For every  $\delta > 0$  satisfying  $(1 - \gamma_1) + 3(1 - \gamma_2) + 6\delta < 1$  there is  $\delta' > 0$  such that*

$$\sum_{n \in \mathbf{B}_N} \psi(n)^{-1} e^{2\pi i \xi n} = \sum_{n=1}^N e^{2\pi i \xi n} + \mathcal{O}(N^{1-\delta-\delta'}).$$

*The implied constant is independent of  $\xi$  and  $N$ .*

*Proof.* For  $N \in \mathbb{N}$  and  $\xi \in \mathbb{T}$  we set

$$S_N(\xi) = \sum_{k \in \mathbf{B}_N} e^{2\pi i \xi k}.$$

Then, by the summation by parts we have

$$\begin{aligned} \sum_{n \in \mathbf{B}_N} \psi(n)^{-1} e^{2\pi i \xi n} &= \sum_{n=1}^N \psi(n)^{-1} (S_n(\xi) - S_{n-1}(\xi)) \\ (19) \quad &= \psi(N+1)^{-1} S_N(\xi) + \sum_{n=1}^N (\psi(n)^{-1} - \psi(n+1)^{-1}) S_n(\xi). \end{aligned}$$

Similarly, we may write

$$(20) \quad \sum_{n=1}^N e^{2\pi i \xi n} = \psi(N+1)^{-1} \sum_{n=1}^N \psi(n) e^{2\pi i \xi n} + \sum_{n=1}^N (\psi(n)^{-1} - \psi(n+1)^{-1}) \sum_{k=1}^n \psi(k) e^{2\pi i \xi k}.$$

Thus, subtracting (20) from (19) we may estimate

$$\left| \sum_{n \in \mathbf{B}_N} \psi(n)^{-1} e^{2\pi i \xi n} - \sum_{n=1}^N e^{2\pi i \xi n} \right| \leq \sum_{n=1}^N |\psi(n)^{-1} - \psi(n+1)^{-1}| \cdot \left| S_n(\xi) - \sum_{k=1}^n \psi(k) e^{2\pi i \xi k} \right| \\ + \psi(N+1)^{-1} \left| S_N(\xi) - \sum_{k=1}^N \psi(k) e^{2\pi i \xi k} \right|.$$

By (10), for any  $\delta > 0$  satisfying  $3(1 - \gamma_1) + (1 - \gamma_2) + 6\delta < 1$  there is  $\delta' > 0$  such that for all  $n \in \mathbb{N}$

$$\left| S_n(\xi) - \sum_{k=1}^n \psi(k) e^{2\pi i \xi k} \right| \leq C \varphi_2(n) n^{-\delta-\delta'}.$$

Using (6) together with [9, Lemma 2.14] we obtain

$$\psi'(n) \lesssim \varphi_2''(n) \lesssim \frac{\varphi_2(n)}{n^2}.$$

Therefore, again by (6) and the monotonicity of  $\varphi_2$  we get

$$|\psi(n)^{-1} - \psi(n+1)^{-1}| \lesssim \sup_{t \in [n, n+1]} |\psi(t)^{-2} \psi'(t)| \lesssim \varphi_2(n)^{-1}.$$

Hence,

$$\sum_{n=1}^N |\psi(n)^{-1} - \psi(n+1)^{-1}| \cdot \left| S_n(\xi) - \sum_{k=1}^n \psi(k) e^{2\pi i \xi k} \right| \lesssim \sum_{n=1}^N n^{-\delta-\delta'} \lesssim N^{1-\delta-\delta'}. \quad \square$$

*Proof of Proposition 3.1.* The  $TT^*$  argument will be critical in the proof. Firstly, let us calculate  $T_N^*$ . By Plancherel's theorem we have

$$\langle T_N f, g \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \mathcal{F}(f \mu_N)(\xi) \overline{\mathcal{F}(g)(\xi)} d\xi = \sum_{n \in \mathbb{Z}} f(n) \overline{\hat{g}(n)} \mu_N(n) = \langle f, T_N^* g \rangle_{L^2(\mathbf{B}_N, \mu_N)}.$$

Therefore, the adjoint operator  $T_N^* : \mathcal{C}(\mathbb{T})^* \rightarrow \mathcal{C}(\mathbf{B}_N)^* = \mathcal{C}(\mathbf{B}_N)$  is given by

$$T_N^*(g) = \hat{g} \cdot \mathbf{1}_{\mathbf{B}_N},$$

and consequently,  $T_N T_N^* : \mathcal{C}(\mathbb{T})^* \rightarrow \mathcal{C}(\mathbb{T})^*$  may be written as

$$T_N T_N^* f = f * \mathcal{F}(\mu_N).$$

Let us observe that it is enough to show

$$(21) \quad \|T_N T_N^*\|_{L^{p'}(\mathbb{T}) \rightarrow L^p(\mathbb{T})} \leq C_p N^{-2/p}.$$

Indeed, for  $f \in L^2(\mathbf{B}_N, \mu_N)$  and  $g \in L^{p'}(\mathbb{T})$  we have

$$|\langle T_N f, g \rangle_{L^2(\mathbb{T})}| = |\langle f, T_N^* g \rangle_{L^2(\mathbf{B}_N, \mu_N)}| \leq \|f\|_{L^2(\mathbf{B}_N, \mu_N)} \|T_N^* g\|_{L^2(\mathbf{B}_N, \mu_N)}$$

and since

$$\|T_N^* g\|_{L^2(\mathbf{B}_N, \mu_N)}^2 = \langle T_N T_N^* g, g \rangle_{L^2(\mathbb{T})} \leq \|T_N T_N^*\|_{L^{p'}(\mathbb{T}) \rightarrow L^p(\mathbb{T})} \|g\|_{L^{p'}(\mathbb{T})}^2,$$

we obtain

$$\|T_N f\|_{L^p(\mathbb{T})} \leq \|T_N T_N^*\|_{L^{p'}(\mathbb{T}) \rightarrow L^p(\mathbb{T})}^{1/2} \|f\|_{L^2(\mathbf{B}_N, \mu_N)}.$$

For the proof of (21), for  $N \in \mathbb{N}$ , let us introduce an auxiliary measure  $\nu_N$  on  $\mathbb{Z}$  and the corresponding linear operator  $S_N : \mathcal{C}(\mathbb{N}_N) \rightarrow \mathcal{C}(\mathbb{T})$ , by setting

$$\nu_N(x) = N^{-1} \sum_{n \in \mathbb{N}_N} \delta_n(x),$$

and

$$S_N f = \mathcal{F}(f \nu_N).$$

Here,  $\mathbb{N}_N := \mathbb{N} \cap [1, N]$ . Reasoning similar to the above applied to the operator  $S_N$  leads to

$$S_N S_N^* f = f * \mathcal{F}(\nu_N).$$

Since  $L^p(\mathbb{T})$  can be embedded into  $\mathcal{C}(\mathbb{T})^*$  for any  $p \geq 1$  we may consider the operators  $T_N T_N^*$  and  $S_N S_N^*$  as mappings on  $L^p(\mathbb{T})$  spaces. Next, we write

$$\begin{aligned} \|T_N T_N^* f\|_{L^p(\mathbb{T})} &= \|f * \mathcal{F}(\mu_N)\|_{L^p(\mathbb{T})} \\ &\leq \|f * \mathcal{F}(\nu_N)\|_{L^p(\mathbb{T})} + \|f * \mathcal{F}(\mu_N - \nu_N)\|_{L^p(\mathbb{T})}. \end{aligned}$$

We are going to show that for each  $p$  satisfying (18) there is  $C_p > 0$  such that

$$(22) \quad \|f * \mathcal{F}(\nu_N)\|_{L^p(\mathbb{T})} \leq C_p N^{-2/p} \|f\|_{L^{p'}(\mathbb{T})},$$

$$(23) \quad \|f * \mathcal{F}(\mu_N - \nu_N)\|_{L^p(\mathbb{T})} \leq C_p N^{-2/p} \|f\|_{L^{p'}(\mathbb{T})}$$

for all  $f \in L^{p'}(\mathbb{T})$ . We start by proving (22) for  $p = 2$ . By Plancherel's theorem we have

$$\begin{aligned} \|f * \mathcal{F}(\nu_N)\|_{L^2(\mathbb{T})} &= \|\hat{f} \nu_N\|_{\ell^2(\mathbb{Z})} \leq \|\nu_N\|_{\ell^\infty(\mathbb{Z})} \|f\|_{L^2(\mathbb{T})} \\ &\leq N^{-1} \|f\|_{L^2(\mathbb{T})}. \end{aligned}$$

On the other hand, for  $p = \infty$  we may write

$$\|f * \mathcal{F}(\nu_N)\|_{L^\infty(\mathbb{T})} \leq \|\mathcal{F}(\nu_N)\|_{L^\infty(\mathbb{T})} \|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})}.$$

Therefore, for  $p \geq 2$  we use Riesz–Thorin interpolation theorem to obtain (22). To show (23), we apply analogous reasoning. Firstly, by Plancherel's theorem we have

$$\begin{aligned} \|f * \mathcal{F}(\mu_N - \nu_N)\|_{L^2(\mathbb{T})} &= \|\hat{f}(\mu_N - \nu_N)\|_{L^2(\mathbb{T})} \leq \|\mu_N - \nu_N\|_{\ell^\infty(\mathbb{Z})} \|f\|_{L^2(\mathbb{T})} \\ &\leq \varphi_2(N)^{-1} \|f\|_{L^2(\mathbb{T})}. \end{aligned}$$

Secondly, for  $p = \infty$  we get

$$\begin{aligned} \|f * \mathcal{F}(\mu_N - \nu_N)\|_{L^\infty(\mathbb{T})} &\leq \|\mathcal{F}(\mu_N) - \mathcal{F}(\nu_N)\|_{L^\infty(\mathbb{T})} \|f\|_{L^1(\mathbb{T})} \\ &\leq N^{-\delta-\delta'} \|f\|_{L^1(\mathbb{T})} \end{aligned}$$

where in the last estimate we have used Lemma 3.2. Thus, again by Riesz–Thorin interpolation theorem, for  $p \geq 2$  we get

$$\begin{aligned} \|f * \mathcal{F}(\mu_N - \nu_N)\|_{L^p(\mathbb{T})} &\leq \|\mu_N - \nu_N\|_{\ell^\infty(\mathbb{Z})}^{2/p} \cdot \|\mathcal{F}(\mu_N - \nu_N)\|_{L^\infty(\mathbb{T})}^{1-2/p} \cdot \|f\|_{L^{p'}(\mathbb{T})} \\ &\lesssim \varphi_2(N)^{-2/p} N^{-(\delta+\delta')(1-2/p)} \|f\|_{L^{p'}(\mathbb{T})}. \end{aligned}$$

Let us recall that for any  $\varepsilon > 0$

$$\varphi_2(N) \gtrsim_\varepsilon N^{\gamma_2 - \varepsilon}$$

Therefore, for the inequality (23) to hold true, we need to have  $\varepsilon > 0$  and  $p > 2$  to satisfy

$$-2(\gamma_2 - \varepsilon)/p - (\delta + \delta')(1 - 2/p) \leq -2/p.$$

Hence,

$$\varepsilon \leq -(1 - \gamma_2) + (\delta + \delta')(p - 2)/2.$$

Because the right hand side has to be positive, we obtain the condition

$$(\delta + \delta')(p - 2)/2 - (1 - \gamma_2) > 0,$$

which is equivalent to

$$p > 2 + 2(1 - \gamma_2)/(\delta + \delta').$$

Since  $3(1 - \gamma_2) + (1 - \gamma_1) + 6\delta < 1$  we conclude

$$p \geq 2 + \frac{12(1 - \gamma_2)}{\gamma_1 + 3\gamma_2 - 3}. \quad \square$$

Next, we show Theorem 1 and Theorem 2.

*Proof of Theorem 1 and Theorem 2.* Let  $(a_n : n \in \mathbb{N})$  be a sequence of complex numbers such that  $\sup_{n \in \mathbb{N}} |a_n| \leq 1$ . Using Proposition 3.1 with  $f(n) = a_n \psi(n)$  we get

$$\int_{\mathbb{T}} \left| \sum_{n \in \mathbf{B}_N} f(n) \psi(n)^{-1} e^{2\pi i \xi n} \right|^p d\xi \lesssim_p N^{p/2-1} \left( \sum_{n \in \mathbf{B}_N} |f(n)|^2 \psi(n)^{-1} \right)^{p/2},$$

thus, by (11),

$$\int_{\mathbb{T}} \left| \sum_{n \in \mathbf{B}_N} a_n e^{2\pi i \xi n} \right|^p d\xi \lesssim_p N^{p/2-1} \left( \sum_{n \in \mathbf{B}_N} \psi(n) \right)^{p/2} \lesssim_p N^{-1} \varphi_2(N)^p.$$

Finally, we may estimate

$$\int_{\mathbb{T}} \left| \sum_{n \in \mathbf{B}_N} e^{2\pi i \xi n} \right|^p d\xi \gtrsim \int_{|\xi| \leq 1/(100N)} \left| \sum_{n \in \mathbf{B}_N} e^{2\pi i \xi n} \right|^p d\xi \gtrsim N^{-1} \varphi_2(N)^p.$$

This completes the proof.  $\square$

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